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Acoustic Signal Detection by Simple Correlators in the Presence of Nongaussian Noise. I. Signal to Noise Ratios and Canonical Forms

by

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ABSTRACT

The detection of underwater acoustic signals by simple auto- and cross-correlation receivers in the presence of non-normal, as well as normal background noise is examined on the basis of signal-to-noise ratios calculated from a generalized deflection criterion. Particular attention is devoted to the effects of impulse noise and mixtures of impulse and normal noise on system performance. Comparisons between system behavior vis-à-vis the two types of interference are made. For impulse noise equivalent in spectral distribution and average intensity to a gauss noise background it is found that the output signal-to-noise (power) ratios are related by the canonical expression

$$(S/N)_I^2 = \frac{(S/N)_G^2}{1 + (1-\mu)/\Lambda \cdot (S/N)_G^2} \quad , \quad 0 \leq \mu \leq 1,$$

where Λ (≥ 0) is the "impulse factor" and μ is the fraction (in average intensity) of the total noise background that is attributable to normal noise. Impulse noise always degrades system performance vis-à-vis normal noise in the autocorrelation reception of stochastic signals, characteristic of applications where passive receiving methods must be used. This degradation can be considerable [0(10db or more)] if the noise is highly impulsive (large Λ) and if large values of $(S/N)_{out}^2$ (> 0 db) are required (for high accuracy of decision). On the other hand, when coherent (i. e. deterministic) signals are employed, so that cross-correlation reception is possible, the degradation may be reduced essentially to zero (i. e. $\Lambda \to 0$) under realizable conditions of operation. It is observed for impulsive, as well as normal noise backgrounds, that cross-correlation receivers are linear in their dependence on signal-to-noise ratio,

i. e. $(S/N)^2_{out} \sim (S/N)^2_{in}$ if sufficiently strong injected signals are employed. The analysis is carried out largely in canonical form, so that the general results for $(S/N)^2_{out}$ can be applied to other, special types of non-normal noise backgrounds. Specific relations are included, along with a detailed summary of the principal results, showing the dependence of $(S/N)^2_{out}$ on $(S/N)^2_{in}$, filtering, delay, noise and signal spectra, etc. for weak and strong inputs, little or heavy post-correlation smoothing and for gauss as well as for impulse noise.

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GLOSSARY OF PRINCIPAL SYMBOLS

A_{01}, A_{02}	=	peak amplitudes of sinusoidal signals
A, A_1, A_2	=	impulse amplitudes
$A_4^{(12)}(\tau, x)$	=	fourth-order moment of nonnormal noise
$a_0^2, a_{01}^2, a_{02}^2$	=	input signal-to-noise power ratios
β	=	reciprocal of average impulse duration
$\overline{F_Z^{(12)2}}$	=	mean-square fluctuation
Δf_e	=	width of equivalent rectangular filter
f	=	frequency
γ	=	impulse noise "density"
Γ_0, Γ_∞	=	filter and impulse structure factors
h_T, h	=	weighting functions of smoothing filter
$I_{12}, \hat{I}_{12}$	=	impulse noise terms
$K_S, K_N, K_N^{(12)}$, etc.	=	auto and cross variance functions of signals and noise
$k_S, k_N, k_S^{(12)}$, etc.	=	normalized auto and cross variance functions of signals and noise
$k_u^{(12)}, k_{u^2}^{(12)}$	=	normalized auto and cross-variance function of the basic impulses $u_1, u_2; u_1^2, u_2^2$
λ_T	=	integral of the smoothing filter's weighting function
$\Lambda^{(12)}(\tau)_I$	=	impulse noise factor

$M_Z(t, \tau)$	=	correlator output
μ	=	fraction of background noise intensity attributable to normal noise
$N(t)$	=	background noise wave
ψ_N, ψ_x, ψ_y	=	mean intensities of N, x, y
$\rho_T(t)$	=	autocorrelation functions of truncated smoothing filter
$\rho_{12}^{(x)}, \rho_{ij}^{(xy)}$, etc.	=	normalized autovariance and cross variance functions of x, x and y, etc.
S, S_1, S_2	=	signal waveforms
$(S/N)_I^2$	=	output (S/N) power ratio for impulse noise
$(S/N)_G^2$	=	output (S/N) power ratio for gauss noise
$(S/N)_{out}^2, (S/N)_{in}^2$	=	output and input (S/N) power ratios
T	=	smoothing interval
τ	=	correlation delay
u, u_1, u_2	=	normalized basic impulse waveforms
ω, ω_0	=	angular frequencies
x, y	=	input waveforms to the correlators
$Z(t, \tau)$	=	output of the correlator's multiplier

1. INTRODUCTION

Although normal, or gaussian noise processes are common sources of interference in such communication problems as signal detection and signal extraction, background noise of non-gaussian character also occurs frequently, particularly in underwater acoustics. The aim of the present paper is to study the effects on signal detection of non-normal background noise accompanying possible acoustic signals. The receivers here are postulated to be simple correlation detectors, employing either auto-or cross-correlation techniques. The criterion of performance is given in terms of a suitably defined output signal-to-noise ratio and its dependence on input signal-to-noise ratio, integration time, and post-detection smoothing. A more comprehensive statistical approach, which makes use of generalized likelihood ratios and general n-th order distributions of the received noise and signal process[1], does not yet appear to be technically feasible with non-normal statistics.* However, by restricting the criterion to the partially descriptive concept of signal-to-noise ratio[2], instead of attempting the optimal and statistically complete description based on decision theory methods,[1,*] and by choosing simple correlators involving at most

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[1] Middleton, D., "An Introduction to Statistical Communication Theory", McGraw-Hill, New York (1960), Part IV and especially Chapter 19.

* Under the more restrictive conditions of independent data samples and long observation times, however, some progress has been made in the more general statistical approach, for correlation devices and certain classes of optimum (e. g. likelihood) systems[1a].

[1a] Wolff, S. S., J. B. Thomas, and T. R. Williams, The Polarity-Coincidence Correlator: A Nonparametric Detection Device, IRE Trans. in Information Theory, Vol. IT-8, 5-9, Jan. (1962), see also ref. 4 therein.

[2] Ref. 1, Secs. (5.3-3, 4).

quadratic operations on the input data, we can construct an essentially complete theory at this level of complexity, since at most fourth-order moments of the input signal and noise processes are required. An essential task of the present study, accordingly, is to obtain these fourth-order moments for a variety of appropriate noise and signal models and to evaluate the resulting output signal-to-noise ratios. Following this, we seek a comparison with similar systems operating against otherwise equivalent[*] normal noise backgrounds. In this way we obtain a direct, quantitative measure of the change in expected performance due to the nongaussian character of the interference. Of particular interest here are both weak and strong-signal performance.

The principal components of our model may be briefly summarized; the key features are (1), a deflection criterion[2] specifying output signal-to-noise ratios; (2), simple (i. e. unweighted) auto-and cross-correlators; (3), finite-time smoothing of the output of the correlator, with suitable low-pass filters; and (4), non-normal, as well as normal noise and signal processes; deterministic signals are also considered. Although the general formulation is made initially in terms of nonstationary processes, specific examples are evaluated for the stationary situation in some detail. Previous work has either focused on the correlation aspects, with and without final smoothing and with normal noise backgrounds[3, 4, 5, 6] or has considered the statistics of the output of more general,

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[*]Equivalent in intensity spectra and mean total intensities; see Eq. (4.4) et seq.

[3]Fano, R. M., "Signal-to-Noise Ratio in Correlation Detectors", Tech. Rept. No. 186, Feb. 19 (1951) MIT Research Laboratory of Electronics.

[4]Faran, J. J. and R. Hills, "Correlators for Signal Reception", Tech. Mem. No. 27, Sept. 15 (1952), Acoustics Research Laboratory(Harvard Univ.)

[5]Faran, J. J. and R. Hills, "The Application of Correlation Techniques to Acoustic Receiving Systems", Tech. Memo No. 28, Nov. 1 (1952), Acoustics Research Laboratory, Div. Appl. Sci., Harvard Univ., Cambridge, Mass. (Sec. 5 for multiple element arrays.)

[6]Lee, Y. W., T. P. Cheatham and J. B. Wiesner, "Application of Correlation Analysis to the Detection of Periodic Signals in Noise, Proc IRE **38**, 1165(1950).

zero-memory (and zero delay) nonlinear devices with non-guassian inputs,[7] or has concentrated on the fourth-moment statistics of various types of non-normal noise[8]. None to date has combined the various elements [(1) - (4) above], in particular, the combination of signals and non-normal noise backgrounds in correlation detectors as presented here, including the role of finite-time averages[9]. Another new feature of the present effort is the development of canonical results for system performance, results and relations that are invariant of specific spectral properties, the random or deterministic natures of signal and noise, and in many cases, even of the specific statistics of the signal and noise processes themselves.

We shall consider in various combinations the following varieties of signals and noise:

Noise Backgrounds:

(1) gaussian noise, as a standard of reference;

(2) low "density" and arbitrary "density" impulse noise (Poisson statistics);*

(3) mixtures, in arbitrary proportions of (1) and (2);

(4) nearly normal noise (as a special case of (3)).

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[7] Mullen, J. A., and D. Middleton, The Rectification of Non-gaussian Noise, Q. App. Math 15, 395 (1958).

[8] Magness, J. A., Spectral Response of a Quadratic Device to Non-gaussian Noise, J. App. Phys, 25, 1357 (1954).

[9] Davenport, W. B., R. A. Johnson and D. Middleton, Statistical Errors in Measurements on Random Time Functions, J. App. Phys. 23, 377(1952); Also, Sec. 16.1-1, 2, 3, ref. 1.

*Section 112., Ref. 1.

Input Signals:

(1) simple sinusoids;

(2) stochastic signals, e.g., normal and impulse processes

The present analysis is restricted to single-element arrays whose performance may be regarded as quantitatively typical of more involved array structures. The effects of multiple-element arrays[5, 10] and optimum linear filtering[*] before correlation, as well as other types of non-normal noise and signal processes, are reserved for a subsequent paper.

Although our treatment is specifically directed to problems of acoustic signal detection, the results may be carried over directly, or with obvious modifications, to analogous situations involving electromagnetic propagation, e.g., radio, radar, etc. Section 2 following presents a general formulation of correlation detection, with the calculation of output signal-to-noise ratios $(S/N)^2_{out}$ based on a deflection criterion; Section 3 gives canonical expressions for $(S/N)^2_{out}$, while Section 4 applies these results to (a), gauss noise backgrounds, (b), impulse noise; (c), mixtures of normal and impulsive noise backgrounds. Section 5 considers limiting forms of the impulse factor $\Lambda^{(12)}(\tau)$, while Section 6 is devoted to the canonical limits of the correlation detectors' outputs for weak and strong signals subject to the various background noises of Section 4. A summary and discussion of the principal results in Section 7 completes the paper. Special relations, needed in the body of the work, are derived in the Appendixes.

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[10] Heaps, H. S., General Theory for the Synthesis of Hydrophone Arrays, J. Acous. Soc. Amer. **32**, 356 (1960). This paper also contains an extensive bibliography of array studies.

[*] Sec. 16.3, ref. 1.

2. FORMULATION

The general structure of the correlation receivers assumed here is shown in Figure 1. The output of A, directly following the zero-memory multiplier is

$$Z(t, \tau) = x(t)\, y\,(t-\tau) \tag{2.1}$$

where x and y are the inputs at a, a' (after possible linear filtering, not however explicitly considered at this point). Specifically, we have

$$x(t) = S^{(1)}(t) + N^{(1)}(t); \quad y(t) = S^{(2)}(t) + N^{(2)}(t), \tag{2.2}$$

in which $N^{(1)}$, $N^{(2)}$ may be correlated, as may be $S^{(1)}$, $S^{(2)}$; however, without any real loss of applicability we assume that S and N are statistically independent (at least through their fourth-order moments). When switch 1 is closed, with

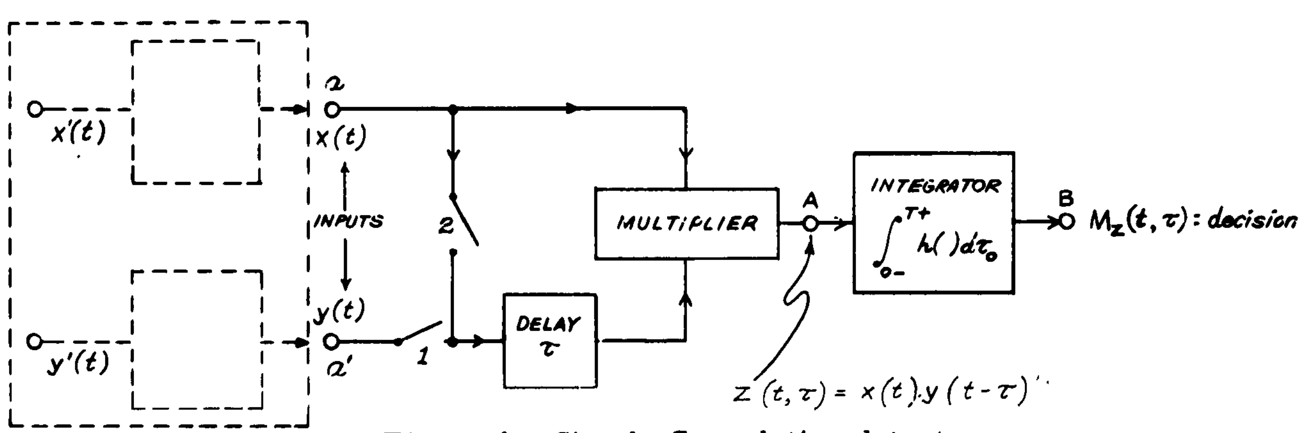

Figure 1. Simple Correlation detectors

switch 2 open, we have a cross-correlator, while when switch 1 is open and 2 is shut, we have an auto-correlator, with $y(t-\tau) = x(t-\tau)$. The output at B at

time t = T, when a decision is to be made, is simply*

$$M_Z(T,\tau) = \int_{0-}^{T+} h(\tau_o)\, x(T-\tau_o)\, y(T-\tau_o-\tau)\, d\tau_o . \tag{2.3}$$

As we shall see below, it is the appropriate first and second moments of M_Z that are needed in our criterion of system performance.

This criterion of performance is a general form of the deflection criterion**, here specifically the ratio of the mean shift in receiver output M_Z when a signal is present vis-à-vis that of noise alone, to the rms fluctuation of this output. The former represents a more realistic version of output signal than normally used in the past[3,4], albeit still a somewhat arbitrary and certainly incomplete statistical description, while the latter is a measure of the interfering background against which the desired signal is perceived. We have accordingly for readouts at time t = T (after smoothing starts at t = 0, cf. (2.3)).

$$\left(\frac{S}{N}\right)^2_{out} = \frac{\left[\,\overline{M_Z(T,\tau)}_{S+N} - \overline{M_Z(t,\tau)}_N\,\right]^2}{\left.\overline{F_Z^{(12)}(T,\tau)^2_{S+N}}\right|_{(nxn)+(sxn)}} , \tag{2.4}$$

with

$$\left.\overline{F_Z^{(12)}(T,\tau)^2_{S+N}}\right|_{(nxn)+(sxn)} \equiv \left\{ \overline{M_Z(T,\tau)^2_{S+N}} - \overline{M_Z(T,\tau)}^2_{S+N} \right\}_{(nxn)+(sxn)} , \tag{2.5a}$$

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*Sec. 16.1-1, of ref. 1.

**See Sec. 5.3-4 of ref. 1.

the mean-square background noise fluctuation. From (2.3) in (2.4), (2.5a) we see that

$$\overline{M_Z(T,\tau)^2_{S+N}} = \int_{0-}^{T+} h(\tau_o)\ \overline{x(T-\tau_o)\ y(T-\tau-\tau_o)}\ d\tau_o\ , \tag{2.5b}$$

$$\overline{M_Z(T,\tau)^2_{S+N}} = \iint_{0-}^{T+} h(\tau_1)\ h(\tau_2)\ \overline{x(T-\tau_1)x(T-\tau_2)y(T-\tau-\tau_1)y(T-\tau-\tau_2)}\ d\tau_1 d\tau_2 \tag{2.5c}$$

where for $M_Z(T,\tau)_N$, etc. we simply set $S^{(1)}, S^{(2)} = 0$ in x and y, (2.2) above*. From (2.5b, c) in (2.4) (2.5a) is at once clear that fourth-order moments of the input processes (x, y) play a critical and determining role in the present theory.

We remark here that the background fluctuation effect in (2.4) is, <u>for detection,</u> usually dependent on both the (nxn) and (sxn) noise products** generated in the nonlinear operation of multiplication. Note that this definition of $(S/N)^2_{out}$ is an extension of the earlier deflection criterion of Lawson and Uhlenbeck[11], in that the (sxn) terms are here included as well. The earlier criterion is more suited, for example, to pulsed signals when one visually compares the S+N state to that of noise alone, the two being juxtaposed, as on a cathode-ray oscilloscope display. The present definition (2.4) is better suited to the distinguishing of a signal <u>in</u> noise, and, of course,

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*Unless otherwise indicated, the superscript bars denote the statistical average over all random components and parameters of x and y; $(\overline{}^{N})$ denotes the average over noise (N) alone, while $(\overline{}^{S})$ similarly indicates the average over S.

**See Ref. 1, sec. 5.1-2.

[11] Lawson, J. L. and G. E. Uhlenbeck, "Threshold Signals", McGraw-Hill (New York), MIT Radiation Laboratory Series No. 24, 1950, sections 7.3, 7.4. See also sec. 5.3-4 of Ref. 1 for an example of (2.4).

its measurement. The two criteria yield the same results in threshold (weak-input signal) operation, as expected since then the (sxn) contribution can be neglected vis-à-vis the dominant (nxn) components, but yield quite different results for strong signals, where the (sxn) terms now predominate. The simpler version may, of course, be obtained directly from our more general results simply by dropping the (sxn) contributions therein, cf. Sec. 6 following.

Besides the auto- and cross-correlation receivers described above and shown in Figure 1, two other forms of correlation reception may also commonly occur. These are (i), an auto-correlation, two-channel receiver (switch 2 open and switch 1 closed), with the *same* signal in each input and independent noises in each channel, so that x and y, Eq. (2.2), become in this instance

$$x(t) = S(t) + N^{(1)}(t); \; y(t) = S(t) + N^{(2)}(t), \tag{2.6}$$

with $N^{(1)}$, $N^{(2)}$ statistically independent. The second system, (ii), involves cross-correlation reception (switch 2 open and switch 1 closed, once more) like that discussed above, but now with $N^{(1)}$, $N^{(2)}$ statistically independent (at least through the fourth-order moments for the present analysis), and, of course, with $S^{(1)}$ and $S^{(2)}$ related so that Eq. (2.2) applies here again. As noted later in Secs. 3 and 4, systems (i) and (ii) are independent of the particular character of the higher (fourth or more) order statistics of the background noise, and so apart from this fact, are not of particular interest to us in the present paper, which is principally concerned with the important cases of statistically related noise backgrounds ($N^{(1)} = N^{(2)}$ in autocorrelation, $\overline{N^{(1)}N^{(2)}} \neq \overline{N^{(1)}N^{(2)}}$, etc. in cross-correlation) and the effects of the non-normality of such noise.

3. CANONICAL FORMS OF $(S/N)^2_{out}$

Our first task, accordingly, is to evaluate the numerator and demoninator of (2. 4) for general noise and signal processes. Let us begin with the fourth-order moment

$$\overline{x_1x_2y_3y_4} = \overline{(S_1^{(1)} + N_1^{(1)})\,(S_2^{(1)} + N_2^{(1)})\,(S_3^{(2)} + N_3^{(2)})\,(S_4^{(2)} + N_4^{(2)})}, \qquad (3.1)$$

where $N_1 = N(t_1)$, etc. and S and N are assumed to be statistically independent. The development of (3. 1) is

$$\begin{aligned}\overline{x_1x_2y_3y_4} = {} & \overline{N_1^{(1)}N_2^{(1)}N_3^{(2)}N_4^{(2)}} + \Big\{ \overline{S_1^{(1)}} \cdot \overline{N_2^{(1)}N_3^{(2)}N_4^{(2)}} + \overline{S_2^{(1)}} \cdot \overline{N_1^{(1)}N_3^{(2)}N_4^{(2)}} \\ & + \overline{S_3^{(2)}} \cdot \overline{N_1^{(1)}N_2^{(1)}N_4^{(2)}} + \overline{S_4^{(2)}} \cdot \overline{N_1^{(1)}N_2^{(1)}N_3^{(2)}} \Big\} \\ & + \Big\{ \overline{S_1^{(1)}S_2^{(1)}} \cdot \overline{N_3^{(2)}N_4^{(2)}} + \overline{S_1^{(1)}S_3^{(2)}} \cdot \overline{N_2^{(1)}N_4^{(2)}} + \overline{S_1^{(1)}S_4^{(2)}} \cdot \overline{N_2^{(1)}N_3^{(2)}} \\ & + \overline{S_2^{(1)}S_3^{(2)}} \cdot \overline{N_1^{(1)}N_4^{(2)}} + \overline{S_2^{(1)}S_4^{(2)}} \cdot \overline{N_1^{(1)}N_3^{(2)}} + \overline{S_3^{(2)}S_4^{(2)}} \cdot \overline{N_1^{(1)}N_2^{(1)}} \\ & + \overline{S_1^{(1)}S_2^{(1)}S_3^{(2)}} \cdot \overline{N_4^{(2)}} + \overline{S_1^{(1)}S_3^{(2)}S_4^{(2)}} \cdot \overline{N_2^{(1)}} + \overline{S_1^{(1)}S_2^{(1)}S_4^{(1)}} \cdot \overline{N_3^{(2)}} \\ & + \overline{S_2^{(1)}S_3^{(2)}S_4^{(2)}} \cdot \overline{N_1^{(1)}} \Big\} \\ & + \overline{S_1^{(1)}S_2^{(1)}S_3^{(2)}S_4^{(2)}}. \end{aligned} \qquad (3.2)$$

The first term of (3.2) represents (nxn) components, the next three, in { }, are (sxn) modulation products, while the last term denotes the (sxs) products. Now when $\overline{N_i} = 0$ and $\overline{N_i N_j N_k} = 0$ and/or $\overline{S_i} = \overline{S_i S_j S_k} = 0$, (3.2) reduces to

$$\overline{x_1 x_2 y_3 y_4} = \overline{N_1^{(1)} N_2^{(1)} N_3^{(2)} N_4^{(2)}} + \left\{ K_{S-12}^{(1)} K_{N-34}^{(2)} + K_{S-12}^{(12)} K_{N-24}^{(12)} + K_{S-14}^{(12)} K_{N-23}^{(12)} + K_{S-23}^{(12)} K_{N-14}^{(12)} + K_{S-24}^{(12)} K_{N-13}^{(12)} + K_{S-34}^{(2)} K_{N-12}^{(1)} \right\} + \overline{S_1^{(1)} S_2^{(1)} S_3^{(2)} S_4^{(2)}}, \qquad (3.3)$$

where $K_S^{(1)}$, $K_N^{(1)}$ etc., are the various auto-and cross-variance functions of $S^{(1)}$, $N^{(1)}$, etc., e.g.,

$$K_{S-12}^{(1)} = \overline{S^{(1)}(t_1) S^{(1)}(t_2)} = K_S^{(1)}(t_1, t_2);\ K_{N-24}^{(12)} = \overline{N^{(1)}(t_2) N^{(2)}(t_4)} = K_N^{(12)}(t_2, t_4), \text{ etc.} \qquad (3.4a)$$

which for stationary processes reduce to*

$$K_{S-12}^{(1)} = K_S^{(1)}(t_1 - t_2) = K_S^{(1)}(|t_1 - t_2|);\ K_{N-24}^{(12)}(t_2 - t_4) = K_N^{(21)}(t_4 - t_2), \text{ etc.} \qquad (3.4b)$$

The assumption that $\overline{N} = 0$ and $\overline{N_i N_j N_k} = 0$ is not restrictive for most applications; often, also, $\overline{S_i} = \overline{S_i S_j S_k} = 0$, for narrow-band signals or more general signals without dc components, which is enough to insure that (3.2) reduces to (3.3).

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*Note the convention in the ordering of the subscription on t in K_S, K_N, (3.46): first number on t refers to the first component (S or N), etc.

The signal-to-noise ratio (2.4) can now be put into canonical form, independent of the specific statistics of signal and noise. For the general non-stationary cases (with $\overline{N} = 0$, etc., cf. (3.3) we may write

$$\left(\frac{S}{N}\right)^2_{out} = \left(\int_{0-}^{T+} h(\tau_1) K_{S-12}^{(12)} (T, \tau_1, T-\tau-\tau_1)\, d\tau_1 \right)^2 \Big/ \left. \overline{F_Z^{(12)2}} \right\}_{(nxn)+(sxn)} \qquad (3.5)$$

where from (3.3), (3.4a), we have for the mean-square background noise fluctuation

$$\left. \overline{F_Z^{(12)2}} \right|_{(nxn)+(sxn)} = \iint_{0-}^{T+} h(\tau_1) h(\tau_2) \Big\{ \left[A_4^{(12)}(t_1, \ldots, t_4) - K_N^{(12)}(t_1, t_3)^2 \right]$$

$$+ \Big[K_S^{(1)}(t_1, t_2) K_N^{(2)}(t_3, t_4) + K_S^{(12)}(t_1, t_4)\, K_N^{(12)}(t_2, t_3)$$

$$+ K_S^{(12)}(t_2, t_3) K_N^{(12)}(t_1, t_4) + K_S^{(2)}(t_3, t_4) K_N^{(1)}(t_1, t_2) \Big]_{(sxn)} \Big\} d\tau_1 d\tau_2 \qquad (3.6)$$

with $t_1 = T - \tau_1$; $t_2 = T - \tau_2$; $t_3 = T - \tau - \tau_1$; $t_4 = T - \tau - \tau_2$, and

$$A_4^{(12)}(t_1, \ldots, t_4)_N = \overline{N^{(1)}(t_1) N^{(1)}(t_2) N^{(2)}(t_3) N^{(2)}(t_4)}. \qquad (3.6b)$$

(An expression analogous to (3.6b) for the fourth-order signal (sxs) term is given by $A_4^{(12)}(t_1, \ldots, t_4)_S$, with N replaced by S in (3.6b).

In many applications we can regard the component noise and signal processes as essentially stationary for the observation and smoothing times used. Then (3.5) - (3.6) simplify considerably. To see this, we first let

$$\lambda_T \equiv \int_{0-}^{T+} h(\tau_o)\, d\tau_0 \tag{3.7a}$$

$$\left.\begin{aligned} \rho_T(x) &\equiv \int_{-\infty}^{\infty} h_T(u)\, h_T(u+x)\, du \\ &= 0, \; |x| > T \end{aligned}\right\} \; ; \quad h_T(u) = h(u), \quad 0 < t < T+; \; = 0 \text{ elsewhere} \tag{3.7b}$$

and

$$\therefore \int_{-\infty}^{\infty} \rho_T(x)\, dx = \lambda_T^2 \, . \tag{3.7c}$$

The quantity ρ_T is the auto-correlation function of the final smoothing filter*, cf. Figure 1, for the interval (0-, T+). Then our expressions (3.5), (3.6) for $(S/N)^2_{out}$ reduce in these stationary cases to

$$\left(\frac{S}{N}\right)^2_{out} = \lambda_T^2 \; K_S^{(12)}(\tau)^2 \Big/ \left. \overline{F_Z^{(12)2}} \right|_{(nxn)+(sxn)} \tag{3.8}$$

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*See Ref. 1, sec. 3.3-1.

where now

$$\overline{F_Z^{(12)2}} = \left\{ \int_{-\infty}^{\infty} \rho_T(x) \left[A_4^{(12)}(\tau, x)_N - K_N^{(12)}(\tau)^2 \right] dx \right\}_{(nxn)}$$

$$+ \left\{ \int_{-\infty}^{\infty} \rho_T(x) \left[K_S^{(1)}(x) K_N^{(2)}(x) + K_S^{(2)}(x) K_N^{(1)}(x) + K_S^{(12)}(\tau+x) K_N^{(12)}(\tau-x) \right.\right.$$

$$\left.\left. + K_S^{(12)}(\tau-x) K_N^{(12)}(\tau+x) \right] dx \right\}_{(sxn)}. \tag{3.9}$$

The first term once more represents the (nxn) noise products, while the second represents the (sxn) noise modulation components, arising from the intermixing of signal and noise in the course of multiplication. The fourth-order term $A_4^{(12)}$ is specifically

$$A_4^{(12)}(\tau, x)_N = \overline{N^{(2)}(0) N^{(2)}(\tau) N^{(1)}(\tau) N^{(1)}(\tau+x)} \quad (\neq A_4^{12}(x, \tau)_N), \tag{3.10}$$

with an analogous expression for $A_4^{(12)}(\tau, x)_S$ on replacing N by S. We remark again that (3.3), (3.6a) and (3.9) are derived under the assumption of vanishing signal and noise means and third moments. However, if $\overline{N} \neq 0$, and if $\overline{S} = \overline{S_i S_j S_k} = 0$, then we may still use (3.3), (3.6a), (3.9) but with K_N replaced by M_N, the second-moment function of N, e.g. $M_{N-12} = K_{N-12} + \overline{N}_1 \cdot \overline{N}_2$, etc. If S has not vanished, and neither do the third moments of N and S, the complete expression (3.2) must be employed in the calculation of $\overline{F_Z^{(12)2}}$, as is the case generally when the first and third moments of S and N jointly are not zero.

For correlation reception of the types (i), (ii) described at the end of section 2, we find directly that Eq. (3.10) gives now

$$A_4^{(12)}(\tau, x)_N = \overline{N^{(2)}(0)N^{(2)}(x)} \cdot \overline{N^{(1)}(\tau)N^{(1)}(\tau+x)} = K_N^{(2)}(x)K_N^{(1)}(x), \quad (3.11)$$

while $K_S^{(12)}(\tau) \longrightarrow K_S(\tau)$, since $S^{(1)} = S^{(2)} = S$, and $K_N^{(12)}(\tau) = 0$, from the assumed independence here of the background noise processes. The output noise term, $\overline{F_Z^{(12)2}}$, in (3.8), (3.9) becomes specifically

$$\overline{F_Z^{(12)2}} = \int_{-\infty}^{\infty} \rho_T(x) \left\{ K_N^{(1)}(x)K_N^{(2)}(x) \Big|_{(n\times n)} + K_S(x) \left[K_N^{(1)}(x) + K_N^{(2)}(x)\right] \right\} dx \quad (3.12)$$

From (3.11) it is immediately clear that only second-order moments of signal and noise are now involved in determining $(S/N)^2_{out}$, (3.8), unlike $(S/N)^2_{out}$ for the correlation receivers principally considered in this paper.

4. OUTPUT SIGNAL-TO-NOISE RATIO FOR GAUSS AND IMPULSE NOISE AND SIGNAL PROCESSES

Let us examine $(S/N)^2_{out}$ for the specific cases of gaussian and impulsive noise backgrounds, and also for various deterministic and stochastic signals. Since only the correlation function of the signal appears in (3.8), (3.9), we can still treat $(S/N)^2_{out}$ canonically as far as the signal is concerned, but because of the fourth-order noise term $(A_4^{(12)})$ appearing in the mean-square fluctuation we must now consider the statistical character of the background noise in more detail. We begin with normal noise, which is a natural standard against which to compare system behavior with non-normal input processes. We find directly that* (for stationary processes)

$$\left.\overline{x_1 x_2 y_3 y_4}^{\,NorS}\right|_{gauss} = \psi_x \psi_y \left\{ \rho_{12}^{(x)} \rho_{34}^{(y)} + \rho_{13}^{(xy)} \rho_{24}^{(xy)} + \rho_{23}^{(xy)} \rho_{14}^{(xy)} \right\}, \qquad (4.1)$$

where $\bar{x} = \bar{y} = 0$ and

$$\psi_x = \overline{x^2};\ \psi_y = \overline{y^2};\ \rho_{12}^{(x)} = \overline{x_1 x_2} \Big/ \overline{x^2};\ \rho_{13}^{(xy)} = \overline{x_1 y_3} \Big/ \sqrt{\overline{x^2}\cdot\overline{y^2}},\ \text{etc., and}$$

$$\rho_{ij}^{(xy)} = \rho_{ji}^{(yx)} = \rho^{(xy)}(t_i - t_j). \qquad (4.1a)$$

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*See Eq. 7.29a of Ref. 1.

Consequently, we have

$$\left[A_4^{(12)}(\tau, x)_N - K_N^{(12)}(\tau)^2\right]_{\text{gauss}} = K_N^{(12)}(\tau+x)K_N^{(12)}(\tau-x) + K_N^{(1)}(x)K_N^{(2)}(x). \tag{4.2}$$

For non-normal processes, however, we cannot expect the fourth-order moments to factor like (4.1) for gauss processes. For the non-normal noise considered specifically here, namely impulse or Poisson noise, we obtain from Appendix I

$$\left[A_4^{(12)}(\tau, x)_N - K_N^{(12)}(\tau)^2\right]_{\text{imp}} = I_{12}(\tau, x) + \left\{K_N^{(12)}(\tau+x)K_N^{(12)}(\tau-x) + K_N^{(1)}(x)\, K_N^{(2)}(x)\right\}_{\text{imp}} \tag{4.3}$$

where

$$I_{12}(\tau, x) = (\beta/\gamma)\int_{-\infty}^{\infty} \overline{(A_1^2\,\gamma)\,(A_2^2\gamma)\, u_1(\tau_o+\tau)u_1(\tau_o+\tau+x)u_2(\tau_o)u_2(\tau_o+x)}\; d\tau_o \tag{4.3a}$$

in which β is the reciprocal of the average duration of a typical impulse and γ is the impulse "density" = (av. number of impulses/sec) x (mean duration of a typical impulse). Here u_1, u_2 represent two, possibly different classes of basic, normalized impulse waveforms which may or may not be generated by a common stochastic mechanism, while $A_1\sqrt{\gamma}$, etc., are suitably normalized impulse amplitudes, which may or may not be random. (See Appendix I for details.) It is assumed again that the first and third-order moments of N_{imp} vanish also. In particular, the normalizations on $A_{1,2}$ etc. used here are such

that $A_{1,2}\sqrt{\gamma} = 0\ (\gamma^0)$, so that $K_N^{(1)}(\tau)_{imp} \longrightarrow K_N^{(1)}(\tau)_{gauss}$ as $\gamma \to \infty$, since as $\gamma \to \infty$ this Poisson process becomes normal[12], and $\lim_{\gamma \to \infty} I_{12}(\tau, x) \to 0$, accordingly, cf. (4.3a).

Next, let us use the model employing normal noise backgrounds as our reference against which to compare the effects of other non-normal noises. For this we set the spectrum and intensity of the latter equal to that of the reference gauss process, e.g.*

$$K_N^{(12)}(\tau)_{imp} = K_N^{(12)}(\tau)_{gauss};\ K_N^{(1)}(\tau)_{imp} = K_N^{(1)}(\tau)_{gauss}, \text{ etc.,} \tag{4.4}$$

and this is achieved physically by choosing for the fundamental mechanism of the standard gauss noise the same basic impulses as in the impulse (e.g. Poisson) noise model, except that $\gamma_{gauss} \longrightarrow \infty$ now. Comparing (4.2) and (4.3) subject to (4.4) shows us at once that $I_{12}(\tau, x)$ represents the effect (peculiar to the impulse noise model) on the critical fourth-order moment when standardized to the gauss for comparison. Accordingly, (3.9) can be written

$$\left.\overline{F_Z^{(12)2}}\right|_{imp} = \left.\overline{F_Z^{(12)2}}\right|_{gauss} + \int_{-\infty}^{\infty} \rho_T(x)\, I_{12}(\tau, x)\, dx, \tag{4.5}$$

---

[12] Middleton, D., On The Theory of Random Noise; Phenomenlogical Models I, II, J. App. Phys. 22, 1143, 1153, 1326 (1951). See also sections 11.2-2,3 of ref. 1.

*For non-vanishing noice dc, but $\overline{S} = \overline{S_i S_j S_k} = 0$ we have $M^{(12)}(\tau)_{imp} = M^{(12)}(\tau)$ gauss, etc. cf. remarks following eq. (3.10).

where

$$\overline{F_Z^{(12)2}}\Bigg|_{gauss} = \int_{-\infty}^{\infty} \rho_T(x) \left[K_N^{(12)}(\tau+x) K_N^{(12)}(\tau-x) + K_N^{(1)}(x) K_N^{(2)}(x) \right] dx$$

$$+ \int_{-\infty}^{\infty} \rho_T(x) \left[K_S^{(1)}(x) K_N^{(2)}(x) + K_S^{(2)}(x)\, K_N^{(1)}\ (x) \right.$$

$$\left. + K_S^{(12)}(\tau+x)\, K_N^{(12)}(\tau-x) + K_S^{(12)}(\tau-x) K_N^{(12)}(\tau+x) \right]_{(sxn)} dx, \qquad (4.6)$$

and where the term containing I_{12} in (4.5) represents (nxn) noise products. With the help of (3.8) and

$$\hat{I}_Z^{(12)}(\tau) \equiv \int_{-\infty}^{\infty} \rho_T(x)\, I_{12}(\tau, x)\, dx \qquad (4.7)$$

we can now express $(S/N)^2_{out-imp}$ as a function of $(S/N)^2_{out-gauss}$. Substituting (4.7) into (4.5) and (4.5) into (3.8), remembering that $I_{12}(\tau)_{gauss} = 0$ (cf. above), (and using Eq. A.1-8 also), we have directly

$$\left(\frac{S}{N}\right)^2_{out-I} = \frac{(S/N)^2_{out-G}}{1 + \Lambda^{(12)}(\tau)_I (S/N)^2_{out-G}}, \qquad (4.8)$$

in which

$$\Lambda^{(12)}(\tau)_I \equiv \hat{I}_Z^{(12)}(\tau) \Big/ \lambda_T^2 \, K_S^{(12)}(\tau)^2$$

$$= a_{01}^{-2}\, a_{02}^{-2}\, k_S^{(12)}(\tau)^{-2} \; \frac{\overline{A_1^2 A_2^2}}{\overline{A_1^2} \cdot \overline{A_2^2}}$$

$$\cdot \frac{1}{\gamma} \left[\frac{\int_{-\infty}^{\infty} \rho_T(x)\, dx \int_{-\infty}^{\infty} \overline{u_1(\lambda+\tau)\, u(\lambda+\tau+x) u_2(\lambda) u_2(\lambda+x)}\, d\lambda}{\beta \int_{-\infty}^{\infty} \rho_T(x)\, dx \int_{-\infty}^{\infty} \overline{u_1^2(\lambda)}\, d\lambda \int_{-\infty}^{\infty} \overline{u_2^2(\lambda)}\, d\lambda} \right] \tag{4.9}$$

where $a_{01}^2 = \overline{S^{(1)2}} \Big/ \overline{N^{(1)2}}$, $a_{02}^2 = \overline{S^{(2)2}} \Big/ \overline{N^{(2)2}}$ are input signal-to-noise (power) ratios; $k_S^{(12)}(\tau)$ is the normalized (cross-correlation) function of $S^{(1)}$ and $S^{(2)}$ with delay τ, viz:

$$k_S^{(12)}(\tau) = K_S^{(12)}(\tau) \Big/ \sqrt{K_S^{(1)}(0) K_S^{(2)}(0)}; \quad K_S^{(12)}(\tau) = \overline{S^{(1)}(\tau)\, S^{(2)}(0)}, \tag{4.9a}$$

for these stationary processes. The bars over u_1, u_2 in (4.3a) and (4.9) indicate statistical averages over possible random parameters in the normalized waveforms of the basic impulses. Here $0 \le \gamma \le \infty$; note that when the pulse "density" γ becomes infinite (the case of gauss noise), $(S/N)^2_{out\text{-}I} = (S/N)^2_{out\text{-}G}$, as required, since then $\Lambda^{(12)}(\tau)_I \to 0$. Observe, also that <u>for fixed $\overline{N^2}$</u>,

$\Lambda^{(12)}(\tau)_I$ becomes infinite as $\gamma \to 0$, i.e. as the pulse density is decreased, individual pulse strength is increased in order to maintain $\overline{N^2}$ at a fixed value ($\neq 0$). For our Poisson impulse noise model it can furthermore be shown (cf. Appendix II) that $\Lambda^{(12)}(\tau)_I$ is always positive (or zero); (we recall, also, that $\bar{u} = 0$ here; there is no dc component in these impulses). Figure 2 shows $(S/N)^2_{out-I}$ vs. $(S/N)^2_{out-G}$ for various values of $\Lambda^{(12)}_I$; a discussion is given in Section 7 following.

Sometimes a more appropriate model of the background interference is one that combines both the gaussian and impulsive noise mechanisms. We now use our preceeding results to compare the performance of these simple correlator systems when a mixture of this type appears, with systems operating against normal noise alone. From (4. 5), (4. 7) and (4. 9) we have

$$\left.\overline{F_Z^{(12)2}}\right|_{imp} = \left.\overline{F_Z^{(12)2}}\right|_{gauss} + \lambda_T^2 K_S^{(12)}(\tau)^2 \Lambda^{(12)}(\tau)_I. \tag{4. 10a}$$

Now setting μ equal to the fraction of total background that is normal, we can write

$$\left.\overline{F_Z^{(12)2}}\right|_{I+G} = (1-\mu)\left.\overline{F_Z^{(12)2}}\right|_{imp} + \mu\, \overline{F_Z^{(12)2}}_{gauss}, \quad 0 \leq \mu \leq 1. \tag{4. 10b}$$

(Equation (4. 10b) defines μ quantitatively in terms of the mean-square background intensities.) From (4. 10) in (4. 11) we get

$$\left.\overline{F_Z^{(12)2}}\right|_{I+G} = \left.\overline{F_Z^{(12)2}}\right|_{gauss} + (1-\mu)\lambda_T^2 K_S^{(12)}(\tau)^2 \Lambda^{(12)}(\tau)_I, \tag{4. 11}$$

and so Eq. (4.8) now takes the more general form for mixtures:

$$\left(\frac{S}{N}\right)^2_{out(I+G)} = \frac{(S/N)^2_{out-G}}{1 + (1-\mu)\ \Lambda^{(12)}(\tau)_I (S/N)^2_{out-G}} \tag{4.12}$$

which shows, not unexpectedly, that "diluting" the impulse noise background with normal noise reduces the effect of the impulsive noise vis-à-vis normal noise alone. Figure 2 shows this explicitly.

Finally, there remains the calculation of explicit results for the second- and fourth-order signal terms, the former appearing in $(S/N)^2_{out}$, Eqs. (2.5), (3.8) and the latter appearing in the fluctuating background under certain conditions (not considered further here). We present a few specific results in the stationary regime:

Sinusoidal Signal: $S^{(1)} = A_{01} \cos \omega_o t;\ S^{(2)} = A_{02} \cos(\omega_o t + \psi_o), \psi_o$ fixed. (4.13)

$$\therefore\ K_S^{(12)}(\tau) = \frac{A_{01}A_{02}}{2} \cos(\omega_o \tau + \psi_o), \tag{4.14}$$

and

$$\overline{S_1^{(1)} S_2^{(1)} S_3^{(2)} S_4^{(2)}} = \frac{A_{01}^2 A_{02}^2}{8} \Big\{ \cos \omega_o (t_1 + t_2 - t_3 - t_4 - 2\psi / \omega_o) + \cos \omega_o (t_1 - t_2 + t_3 - t_4) + \cos \omega_o (t_1 - t_2 - t_3 + t_4) \Big\}, \tag{4.15}$$

which becomes from (3.10) adapted to signals, as indicated:

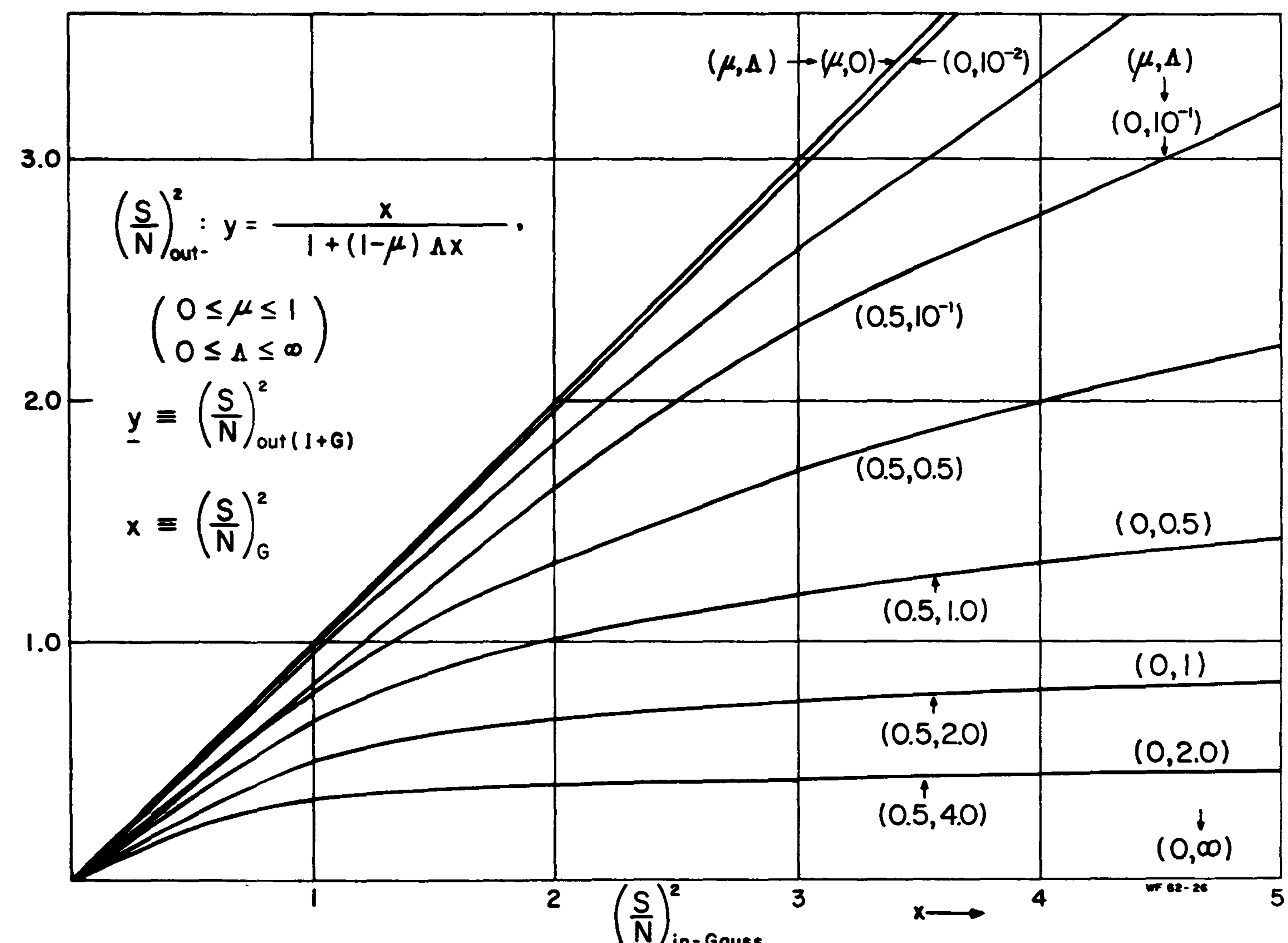

FIGURE 2. Output $\left(\frac{S}{N}\right)^2$ for Impulse Noise ($\mu=0$) and Mixtures of Gauss and Impulse Noise ($0<\mu<1$)

$$A_{14}^{(12)}(\tau, x)_S = \overline{S^{(2)}(0)S^{(2)}(x)S^{(1)}(\tau)S^{(1)}(\tau+x)} = \frac{A_{01}^2 A_{02}^2}{8}\left[\cos 2\omega_o x + 2\cos^2(\omega_o \text{-} \tau - \psi_0)\right] \tag{4.15a}$$

$$\therefore A_{14}^{(12)}(\tau, x) - K_S^{(12)}(\tau)^2 = \frac{A_{01}^2 A_{02}^2}{8}\left[\cos 2\omega_o x + 2\sin\omega_o\tau\sin\psi_o\right], \tag{4.16}$$

which is independent of τ when $\psi_o = 0, \pm \pi, \pm 2\pi, \ldots$, etc. With a gaussian signal, $A_{14}^{(12)}(\tau, x)_S - K_S^{(12)}(\tau)^2$ is given by $K_S^{(12)}(\tau+x)\, K_S^{(12)}(\tau-x) + K_S^{(1)}(x)K_S^{(2)}(x)$, while for impulsive noise signals (4.3) applies with S replacing N.

We remark, also, that for the correlation receivers of types (i), (ii), (cf. end of sec. 2 above) where the background noise is assumed independent between the two input channels, there is no distinction between impulse and gauss noise backgrounds on the basis of signal-to-noise ratio performance, when each has the same mean intensity and spectrum, as is postulated throughout this study. This follows at once from (3.11). Accordingly, the systems of chief interest to us here are those of the main body of the text, wherein the background noise exhibits the fourth-moment behavior (4.3a), etc., leading to (4.9), et seq.

5. LIMITING FORMS OF THE IMPULSE FACTOR $\Lambda^{(12)}(\tau)$:

In what follows (cf. secs. 6, 7) we shall need to consider various limiting forms of the "impulse factor" $\Lambda^{(12)}(\tau)$, Eq. (4.9). The two limiting cases of chief interest are governed (apart from the effect of "impulsiveness" as such" by the degree of smoothing or filtering achieved by the final integrating filter in our simple auto-or-cross-correlators, cf. Figure 1. We distinguish here the opposite situations of (1) heavy smoothing, where a narrow low-pass filter is used, and (2) no smoothing, where the integrating filter is essentially very wide vis-à-vis its input. We find explicitly that for heavy smoothing (3.7b) simplifies to

$$\rho_T(x) \doteq \rho_T(0) + x\rho'_T(0) + \ldots \doteq \rho_T(0) \tag{5.1a}$$

and for no smoothing (3.7b) becomes

$$\rho_T(x) \simeq \delta(x-0), \text{ with } \rho_T(0) = \int_{-\infty}^{\infty} h_T(u)^2 du = 2\Delta f_e, \tag{5.1b}$$

where Δf_e is the width of the equivalent rectangular filter (cf. p. 167, Eq. (3.99), ref. 1). Also, for no smoothing $h_T(u) = \delta(u-0)$; $\lambda_T = 1$ and (3.7c) still applies. Accordingly, Eq. (4.9) assumes the specialized forms:

Heavy Smoothing:

(a) auto-correlator:

$$\Lambda^{(11)}(\tau) \doteq \frac{1}{\gamma} \frac{\overline{A^4}}{\overline{A^2}^2} a_0^{-4} \frac{\overline{k_u(\tau)^2}}{k_S(\tau)^2} \left[\frac{\rho_T(0)}{\lambda_T^2 \beta}\right] \tag{5.2}$$

(b) cross-correlator:

$$\Lambda^{(12)}(\tau) \doteq \frac{1}{\gamma} \frac{\overline{A_1^2 A_2^2}}{\overline{A_1^2} \cdot \overline{A_2^2}} a_{01}^{-2} a_{02}^{-2} \frac{\overline{k_u^{(12)}(\tau)^2}}{k_S^{(12)}(\tau)^2} \left[\frac{\rho_T(0)}{\lambda_T^2 \beta} \right] \tag{5.3}$$

No Smoothing:

(a) auto-correlator:

$$\Lambda^{(11)}(\tau) \simeq \frac{1}{\gamma} \frac{\overline{A^4}}{\overline{A^2}^2} a_0^{-4} \frac{k_{u^2}(\tau)}{k_S(\tau)^2} \left[\frac{1}{\beta} \frac{\int_{-\infty}^{\infty} \overline{u^4(\lambda)}\, d\lambda}{\left(\int_{-\infty}^{\infty} \overline{u^2(\lambda)}\, d\lambda \right)^2} \right] \tag{5.4}$$

(b) cross-correlator:

$$\Lambda^{(12)}(\tau) \simeq \frac{1}{\gamma} \frac{\overline{A_1^2 A_2^2}}{\overline{A_1^2} \cdot \overline{A_2^2}} a_{01}^{-2} a_{02}^{-2} \frac{k_{u^2}^{(12)}(\tau)}{k_S^{(12)}(\tau)^2} \cdot \left[\frac{1}{\beta} \frac{\left(\int_{-\infty}^{\infty} \overline{u_1^4(\lambda)}\, d\lambda \int_{-\infty}^{\infty} \overline{u_2^4(\lambda)}\, d\lambda \right)^{1/2}}{\int_{-\infty}^{\infty} \overline{u_1^2}\, d\lambda \cdot \int_{-\infty}^{\infty} \overline{u_2^2}\, d\lambda} \right] \tag{5.5}$$

Presently, (cf. sec. 6), we shall find it convenient, also, to use the relations

$$\Gamma_0^{(11)} \equiv \overline{A^4}/\overline{A^2}\,; \; \Gamma_0^{(12)} \equiv \overline{A_1^2 A_2^2} \Big/ \overline{A_1^2}\; \overline{A_2^2}\,; \tag{5.6a}$$

$$\Gamma_\infty^{(11)} \equiv \frac{\overline{A^4}}{\overline{A^2}^2} \quad \frac{1}{\beta} \int_{-\infty}^{\infty} \overline{u^4(\lambda)}\, d\lambda \Bigg/ \left(\int_{-\infty}^{\infty} \overline{u^2(\lambda)}\, d\lambda \right)^2 \tag{5.6b}$$

$$\Gamma_\infty^{(12)} \equiv \frac{\overline{A_1^2 A_2^2}}{\overline{A_1^2} \cdot \overline{A_2^2}} \quad \cdot \quad \frac{1}{\beta} \quad \frac{\left(\int_{-\infty}^{\infty} \overline{u_1^4(\lambda)}\, d\lambda \int_{-\infty}^{\infty} \overline{u_2^4(\lambda)}\, d\lambda \right)^{1/2}}{\int_{-\infty}^{\infty} \overline{u_1^2}\, d\lambda \int_{-\infty}^{\infty} \overline{u_2^2}\, d\lambda} \tag{5.6c}$$

which depend on filter and impulse structure alone.

The various normalized covariance functions appearing in (5.2)-(5.5) are specifically

$$\overline{k_{u^2}(\tau)} = \overline{\left(\int_{-\infty}^{\infty} u(\lambda+\tau) u(\lambda)\, d\lambda \right)^2} \Bigg/ \left(\int_{-\infty}^{\infty} \overline{u(\lambda)^2}\, d\lambda \right)^2 \neq k_u(\tau)^2; \tag{5.7}$$

$$\text{with } k_u(\tau) = \int_{-\infty}^{\infty} \overline{u(\lambda+\tau)\, u(\lambda)\, d\lambda} \Bigg/ \int_{-\infty}^{\infty} \overline{u^2(\lambda)}\, d\lambda, \tag{5.7a}$$

and

$$\overline{k_u^{(12)}(\tau)^2} = \overline{\left(\int_{-\infty}^{\infty} u_1(\lambda+\tau)\, u_2(\lambda)\, d\lambda \right)^2} \Bigg/ \int_{-\infty}^{\infty} \overline{u_1^2(\lambda)}\, d\lambda \cdot \int_{-\infty}^{\infty} \overline{u_2^2(\lambda)}\, d\lambda \tag{5.8}$$

$$\text{with } k_u^{(12)}(\tau) = \int_{-\infty}^{\infty} \overline{u_1(\lambda+\tau)\, u_2(\lambda)}\, d\lambda \Bigg/ \left(\int_{-\infty}^{\infty} \overline{u_1^2(\lambda)}\, d\lambda \cdot \int_{-\infty}^{\infty} \overline{u_2^2(\lambda)}\, d\lambda \right)^{1/2} \tag{5.8a}$$

and

$$k_{u^2}(\tau) = \int_{-\infty}^{\infty} \overline{u^2(\lambda+\tau)\, u^2(\lambda)}\, d\lambda \Bigg/ \int_{-\infty}^{\infty} \overline{u^4(\lambda)}\, d\lambda \tag{5.9}$$

$$k_{u^2}^{(12)}(\tau) = \int_{-\infty}^{\infty} \overline{u_1^2(\lambda+\tau) u_2^2(\lambda)}\, d\lambda \Bigg/ \left(\int_{-\infty}^{\infty} \overline{u_1^4(\lambda)}\, d\lambda \cdot \int_{-\infty}^{\infty} \overline{u_2^4(\lambda)}\, d\lambda \right)^{1/2} \neq k_{u^2}^{(21)}(\tau) \tag{5.10}$$

Also, we have $k_S(\tau)$, $k_S^{(12)}(\tau)$, given by (4.9a), the former on setting $S^{(1)} = S^{(2)}$. In all instances, $\overline{k_u^{(12)}(\tau)^2}$, $k_{u^2}(\tau)$, etc. vanish as $\tau \to \infty$, since then u(t) and u(t+τ), etc. do not effectively overlap, cf. (5.7) et seq. By the Schwartz inequality it is easily shown that

$$0 \leq \left| k_u^{(12)}(\tau) \right| \leq 1; \quad -1 \leq k_u, k_S, k_S^{(12)} \leq 1; \; 0 \leq k_{u^2}(\tau),\; k_{u^2}^{(12)}(\tau) \leq 1. \tag{5.11}$$

(However, we observe that $\overline{k_u^{(12)}(\tau)^2} \geq k_u^{(12)}(\tau)^2$ (since $\overline{x^2} > \bar{x}^2$, in general), so that $\overline{k_u^{(12)}(\tau)^2}_{max}$ may be larger than unity.) The factors $(\rho_T(0)/\beta\lambda_T^2)$etc. are filter and impulse "shape factors", depending solely on the detailed structure of the elementary impulses in the case of no filtering, and principally on the filter in the case of heavy smoothing(as well as on mean pulse duration$\sim \beta^{-1}$).

We note from (A. 1-7) that the mean noise intensity is (for N_1 or N_2)

$$\overline{N_{imp}^2} = K_N(0)_{imp} = \beta\, \gamma\, \overline{A_{1,2}^2} \int_{-\infty}^{\infty} \overline{u_{1,2}^2(\lambda)}\, d\lambda. \tag{5.12}$$

Consequently, for fixed noise backgrounds $(\overline{N_{imp}^2} > 0)$, and fixed a_{01}^2, a_{02}^2, as $\gamma \to 0$ (i.e. as the noise becomes more "impulsive"), $\Lambda^{(12)}(\tau)$ becomes infinite. On the other hand, if we relax the requirement of fixed noise intensity (> 0) and let $\gamma \to 0$, $\Lambda^{(12)}(\tau)$ also approaches zero, since $a_{01}^2 \sim \gamma^{-1}$, etc. and then $\Lambda^{(12)}(\tau) = 0(\gamma)$. However, it is the former situation that we are normally concerned with in the present paper.

6. LIMITING CASES OF $(S/N)^2_{out}$:

The strong-and weak-signal performance of our correlation receivers are usually the cases of principal interest. Here we shall consider first operation against gauss noise backgrounds, and from these results and the canonical relations, (4.8), (4.12) then examine receiver performance specifically for mixed gauss and impulse noise. In each case we shall assume either heavy smoothing or no essential smoothing at all (cf. Sec. 5). A summary and comparison of the principal results is given in section 7.

Before going on to special cases let us re-express (3.8) in more explicit form for gauss noise backgrounds. We write first for auto-correlation receivers

$$\left(\frac{S}{N}\right)^2_{G\text{-auto}} = \frac{\lambda_T^2 a_0^4 k_S(\tau)^2}{\int_{-\infty}^{\infty} \rho_T(x) \left[B_{(nxn)}(\tau, x) + a_0^2 B_{(sxn)}(\tau, x)\right] dx}, \tag{6.1}$$

where

$$B_{(nxn)}(\tau, x) = k_N(\tau+x)k_N(\tau-x) + k_N^2(x);\ B_{(sxn)}(\tau, x) = 2k_S(x)k_N(x) + k_S(\tau+x)k_N(\tau-x) + k_S(\tau-x)k_N(\tau+x). \tag{6.1a}$$

The corresponding relations for cross-correlation receivers are

$$\left(\frac{S}{N}\right)^2_{G\text{-cross}} = \lambda_T^2 a_{01}^2 a_{02}^2 k_S^{(12)}(\tau)^2 \Big/ \int_{-\infty}^{\infty} \rho_T(x) \Big[B_{(nxn)}^{(12)} (\tau, x) + a_{01}^2 C_{SN}^{(12)} (x)$$

$$+ a_{02}^2 C_{SN}^{(21)}(x) + a_{01}a_{02} C_{SN}^{(12)} (\tau+x, \tau-x) + a_{01}a_{02}C_{SN}^{(12)}(\tau-x, \tau+x)\Big] \, dx \tag{6.2}$$

with

$$B_{(nxn)}^{(12)}(\tau, x) = k_N^{(12)} (\tau+x)\, k_N^{(12)}(\tau-x) + k_N^{(1)}(x)\, k_N^{(2)} (x),$$

$$C_{SN}^{(12)}(x) = k_S^{(1)}(x)k_N^{(2)}(x); \quad C_{SN}^{(21)} (x) = k_S^{(2)}(x)\, k_N^{(1)} (x), \tag{6.2a}$$

$$C_{SN}^{(12)}(\tau+x, \tau-x) = k_S^{(12)}(\tau+x)\, k_N^{(12)}(\tau-x); \; C_{SN}^{(12)}(\tau-x, \tau+x) = k_S^{(12)}(\tau-x)k_N^{(2)}(\tau+x).$$

We are now ready to consider limiting cases:

6.1 Gauss Noise-Autocorrelation Receivers

Here we summarize results for weak and strong signal reception, with heavy smoothing, or none at all. From (6.1), (6.1a) we have

(weak-signals, $a_0^4 << 1$):

$$\left(\frac{S}{N}\right)^2_{G\text{-auto}} \doteq \frac{\lambda_T^2 a_0^4 k_S^2(\tau)}{\int_{-\infty}^{\infty} \rho_T(x) B_{(nxn)}(\tau, x)dx} \doteq \left. \frac{a_0^4 \lambda_T^2 k_S^2(\tau)}{\rho_T(0) \int_{-\infty}^{\infty} B_{(nxn)}(\tau, x)\, dx} \right]_{\text{heavy smoothing}} \tag{6.3a}$$

$$\doteq \left. \frac{\lambda_T^2 a_0^4 k_S^2(\tau)}{1 + k_N^2(\tau)} \right|_{\text{no smoothing}} \doteq \left(\frac{a_0^4 \lambda_T^2}{2} \right)_{\tau=0} \tag{6.3b}$$

For long delays $\tau \to \infty$ and $\therefore B_{(nxn)}(\tau, x) \to k_N^2(x)$ in (6.3a) above. Note the expected dependence on a_0^4, indicating "signal suppression"[13]. For strong-signals we get

<u>(strong signals $a_0^2 >> 1$):</u>

$$\left.\frac{S}{N}\right|^2_{G\text{-auto}} \simeq \frac{\lambda_T^2 a_0^2 k_S^2(\tau)}{\int_{-\infty}^{\infty} \rho_T(x)\, B_{(sxn)}(\tau, x)dx} = \left.\frac{\lambda_T^2 a_0^2 k_S^2(\tau)}{\rho_T(0) \int_{-\infty}^{\infty} B_{(sxn)}(\tau, x)\, dx}\right]_{\text{heavy smoothing}} \tag{6.4a}$$

$$= \left.\frac{\lambda_T^2 a_0^2 k_S^2(\tau)/2}{1 + k_S(\tau)\, k_N(\tau)}\right|_{\text{no smoothing}} . \tag{6.4b}$$

Observe that when $\tau \to \infty$ in (6.4a), then $B_{(sxn)}(\tau, x) \to 2k_S(x)\, k_N(x)$. Again we have the expected "linear" dependence on a_0^2 as $a_0 \to \infty$, i.e., $(S/N)^2_{out} \approx (S/N)^2_{in}$ here. The noise is now (relatively) suppressed, and these autocorrelation receivers operate essentially as linear elements, rather than quadratic devices, as above (6.3a, b) for weak signals.

6.2 Gauss Noise - Cross-correlation Receivers:

Let us consider next the more involved situation of cross-correlation reception. Using (6.2), (6.2a), and paralleling (6.3a)-(6.4b), designating signal No. 1 as received signal, No. 2 as the locally generated "facsmile" of No. 1, we find that "best" operation, not unexpectedly, occurs if we set

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[13] Ref. I, Sections (5.3-4), p. 285, esp.; see also, Sec. 13.2-1(4), for a detailed discussion of this phenomenon.

$a_{02}^2 \to \infty$ or in practice, at least, make a_{01}^2 / a_{02}^2 as small as possible. Then (6.2) reduces to

<u>(weak signals, $a_{01}^2 << 1$):</u>

$$\left.\left(\frac{S}{N}\right)^2_{G\text{-cross}}\right]_{a_{01}^2 >> 1} \doteq \frac{\lambda_T^2 \, a_{01}^2 \, k_S^{(12)}(\tau)^2}{\int_{-\infty}^{\infty} \rho_T(x)\, k_S^{(2)}(x) k_N^{(11)}(x)\, dx} = \left.\frac{\lambda_T^2 \, a_{01}^2 \, k_S^{(12)}(\tau)^2}{\rho_T(0) \int_{-\infty}^{\infty} k_S^{(2)}(x) k_N^{(1)}(x)\, dx}\right] \text{heavy smoothing} \tag{6.5a}$$

$$\doteq \left.\lambda_T^2 a_{01}^2 \cdot k_S^{(12)}(\tau)^2 \right] \text{no smoothing} \tag{6.5b}$$

where the only significant term in the noise background is the (sxn) component, $C_{SN}^{(21)}(x)$, in (6.2), (6.2a). With heavy smoothing note that this background is governed by the spectral character (or autovariance function) of the injected signal, while for no smoothing, performance depends on $k_S^{(12)}(\tau)^2$ alone (apart from a_{01}^2, etc.). Observe also that the delay τ enters only in the cross-variance of the signals, which permits us to adjust S_2 so that $k_S^{(12)}(\tau) = 1$, by some suitable choice of τ. Threshold performance here, nevertheless, is still controlled by the input signal-to-noise ratio a_{01}^2, but is linear in this ratio, as expected of cross-correlation systems, in contrast with the a_0^4 - dependence of autocorrelation devices in similar threshold experiments. Of course, deterministic signal structures are also required for cross-correlation, unlike

the auto-correlation receiver, which may employ stochastic signals.

For strong input signals (6.2(, (6.2a) become, correspondingly, (strong-signals, $a_{01}^2 >> 1$):

$$\left(\frac{S}{N}\right)^2_{G\text{-cross}} \Bigg]_{\substack{a_{02}^2 \\ a_{01}^2} >> 1} \simeq \frac{\lambda_T^2 \, a_{01}^2 \, k_S^{(12)}(\tau)^2}{\int_{-\infty}^{\infty} \rho_T(x)\, dx \left[\frac{a_{01}^2}{a_{02}^2} \, k_S^{(1)}(x)k_N^{(2)}(x) + k_S^{(2)}(x)k_N^{(1)}(x)\right]} \tag{6.6}$$

$$= \left.\frac{\lambda_T^2 \, a_{01}^2 \, k_S^{(12)}(\tau)^2/\rho_T(0)}{\int_{-\infty}^{\infty} \left[\frac{a_{01}^2}{a_{02}^2} \, k_S^{(1)}(x)k_N^{(2)}(x) + k_S^{(2)}(x)k_N^{(1)}(x)\, dx\right]}\right\} \text{heavy smoothing} \tag{6.6a}$$

$$= \frac{\lambda_T^2 \, a_{01}^2}{\left(1 + \frac{a_{01}^2}{a_{02}^2}\right)} \, k_S^{(12)}(\tau)^2 \Bigg|_{\text{no smoothing}} \tag{6.6b}$$

where now the only significant contributions to the background noise arise from the (sxn) components $C_{SN}^{(12)}(x)$, $C_{SN}^{(21)}(x)$. Note again, the dependence on delay through $k_S^{(12)}(\tau)$ only and if $a_{02}^2 \longrightarrow \infty$, (6.6b) reduces once more to (6.5b) as expected from the characteristic linear behaviors of these cross-correlators.

6.3 Impulse and Mixed Noise Backgrounds:

With (6.1)-(6.6) above for $(S/N)_G^2$ in the important limiting cases, in conjunction with (4.8), we can express $(S/N)_I^2$ explicitly for impulse noise in terms of input signal-to-noise ratio, delay, filter effects, etc. with the help of (5.2)-(5.5) for the impulse factor $\Lambda^{(12)}(\tau)_I$. The principal results, corresponding to the conditions of secs. (6.1), (6.2) above, are listed below:

(weak signals, $a_0^4 \ll 1$):

$$\left(\frac{S}{N}\right)^2_{I\text{-out}} \doteq \frac{a_0^4 \lambda_T^2 k_S^2(\tau) \Big/ \rho_T(0) \int_{-\infty}^{\infty} B_{(nxn)}(\tau, x)\, dx}{1 + \gamma^{-1} \Gamma_0^{(11)} \overline{k_u(\tau)^2} \Big/ \beta \int_{-\infty}^{\infty} B_{(nxn)}(\tau, x)\, dx} \quad \text{heavy smoothing} \tag{6.7a}$$

$$\doteq \frac{a_0^4 \lambda_T^2 k_S^2(\tau) \Big/ \left[1 + k_u^2(\tau)\right]}{1 + \gamma^{-1} \Gamma_\infty^{(11)} k_u^2(\tau) \lambda_T^2 \Big/ \left[1 + k_u^2(\tau)\right]} \quad \text{no smoothing} \tag{6.7b}$$

(strong-signals, $a_0^2 \gg 1$):

$$\left(\frac{S}{N}\right)^2_{I\text{-auto}} \simeq \frac{a_0^2 \lambda_T^2 k_S^2(\tau) \Big/ \rho_T(0) \int_{-\infty}^{\infty} B_{(sxn)}(\tau, x)\, dx}{1 + \gamma^{-1} \Gamma_0^{(11)} \overline{k_u(\tau)^2} \Big/ \beta a_0^2 \int_{-\infty}^{\infty} B_{(sxn)}(\tau, x)\, dx} \quad \text{heavy smoothing} \tag{6.8a}$$

$$\simeq \frac{a_0^2 \lambda_T^2 k_S^2(\tau) \Big/ 2\left[1 + k_S(\tau) k_u(\tau)\right]}{1 + (2\gamma)^{-1} \Gamma_\infty^{(11)} k_u^2(\tau) \lambda_T^2 \Big/ a_0^2 \left[1 + k_S(\tau) k_u(\tau)\right]} \quad \text{no smoothing}$$

(weak signals, $a_{01} << 1$, $a_{02}^2 >> 1$):

$$\left(\frac{S}{N}\right)^2_{I\text{-cross}} \doteq \frac{a_{01}^2 \lambda_T^2 k_S^{(12)}(\tau)^2 \Big/ \rho_T(0) \int_{-\infty}^{\infty} k_S^{(2)}(x) k_N^{(1)}(x)\, dx}{1 + \gamma^{-1} \Gamma_0^{(12)} \overline{k_u^{(12)}(\tau)^2} \Big/ a_{02}^2 \beta \int k_S^{(2)}(x)\, k_N^{(1)}(x) dx} \Bigg|_{\text{heavy smoothing}} \qquad (6.9a)$$

$$\doteq \frac{a_{01}^2 \lambda_T^2 k_S^{(12)}(\tau)^2}{1 + \gamma^{-1} \Gamma_\infty^{(12)} k_{u^2}^{(12)}(\tau)\, \lambda_T^2 \Big/ a_{02}^2} \Bigg|_{\text{no smoothing}} \qquad (6.9b)$$

(strong-signals, $a_{01} >> 1$, $a_{02}^2 >> 1$):

$$\left(\frac{S}{N}\right)^2_{I\text{-cross}} \simeq \frac{a_{01}^2 \lambda_T^2 k_S^{(12)}(\tau)^2 \Big/ \rho_T(0) \int_{-\infty}^{\infty} \left[\frac{a_{01}^2}{a_{02}^2} k_S^{(1)}(x) k_N^{(2)}(x) + k_S^{(2)}(x) k_N^{(1)}(x)\right] dx}{1 + \gamma^{-1} \Gamma_0^{(12)} \overline{k_u^{(12)}(\tau)^2} \Big/ a_{02}^2 \beta \int_{-\infty}^{\infty} \left[\frac{a_{01}^2}{a_{02}^2} k_S^{(1)}(x) k_N^{(1)}(x) + k_S^{(2)}(x) k_N^{(1)}(x)\right] dx}$$

(heavy smoothing) (6.10a)

$$\simeq \frac{a_{01}^2 \lambda_T^2 k_S^{(12)}(\tau)^2 \Big/ (1 + a_{01}^2/a_{02}^2)}{1 + \gamma^{-1} \Gamma_\infty^{(12)} k_{u^2}^{(12)}(\tau) \lambda_T^2 \Big/ a_{02}^2 (1 + a_{01}^2/a_{02}^2)} \Bigg|_{\text{no smoothing}} \qquad (6.10b)$$

For mixed gauss and impulsive noise backgrounds one simply inserts a factor $1 - \mu$ as the coefficient of γ^{-1} in the denominators of (6.7a) -(6.10b).

With the correlation receivers of type (i), (ii), mentioned at the end of sec. 2, we find that Eqs. (6.1), (6.2a) become now

$$(S/N)^2_{\text{auto-2channel}} = \frac{\lambda_T^2 \hat{a}_{01}^2 \hat{a}_{02}^2 k_S(\tau)^2}{\int_{-\infty}^{\infty} \rho_T(x) \left[k_N^{(1)}(x) k_N^{(2)}(x) + \left\{ \hat{a}_{02}^2 k_N^{(2)}(x) + \hat{a}_{01}^2 k_N^{(1)}(x) \right\} k_S(x) \right] dx} \tag{6.11}$$

and

$$(S/N)^2_{\text{cross}} = \frac{\lambda_T^2 \hat{a}_{01}^2 \hat{a}_{02}^2 k_S^{(12)}(\tau)^2}{\int_{-\infty}^{\infty} \rho_T(x) \left[k_N^{(1)}(x) k_N^{(2)}(x) + a_{01}^2 k_N^{(2)}(x)\, k_S^{(1)}(x) + a_{02}^2 k_N^{(1)}(x) k_S^2(x) \right] dx} \tag{6.12}$$

where $\hat{a}_{01}^2 = \psi_S \big/ \psi_{N1}$; $\hat{a}_{02}^2 = \psi_S \big/ \psi_{N2}$, and again $a_{01}^2 = \psi_{S1} \big/ \psi_{N1}$, $a_{02}^2 = \psi_{S2} \big/ \psi_{N2}$.

These relations apply for impulse noise or mixtures of such noise with a gauss background, and in fact, for any (stationary) background noise with the same second-order moments as the reference normal process assumed here. Equations (6.3a, b - 6.5 a, b) are basically unchanged, with obvious modifications in details, which are left to the reader.

7. RESULTS AND CONCLUSIONS

Before we summarize the principal results for impulsive noise backgrounds specifically, some general observations may be made at once from the foregoing. We see that, independent of the particular noise and signal statistics:

(1) With heavy smoothing, i. e., $\rho_T(0) = 2\Delta f_e \rightarrow 0$, corresponding to an indefinitely narrow post-correlation filter, we have $(S/N)^2_{out} \longrightarrow \infty$. As expected, for long enough smoothing periods the output signal-to-noise ratio can be made indefinitely great (as long as the input signal is "on", of course). Conversely, with little or no smoothing, $(S/N)^2_{out}$ remains bounded.

(2) In auto-correlation receivers, weak input signals ($a_0^2 << 1$) lead to $(S/N)^2_{out} = 0(a_0^4)$, the well-known phenomenon of <u>signal suppression</u>[13], while for strong inputs ($a_0^2 >> 1$), $(S/N)^2_{out} = 0(S/N)^2_{in}$, and the system is essentially "linear".

(3) With sufficiently strong injected signals at the receiver, cross-correlation is always "linear" in the input signal-to-noise ratio, i. e. $(S/N)^2_{out} = 0(a_{01}^2)$: there is no signal suppression here.

(4) Auto-correlation systems are worse than, or at best, as good as, cross-correlation systems, in (S/N) performance (for the simple correlator considered here).

While (1)- (4) have been established earlier for normal noise processes, we confirm these results now specifically for impulsive noise backgrounds and mixtures of gauss and impulse noise. When the background noises in the two-channel correlators (types i, ii included) are independent (at least through the fourth-order moments), their evaluation of performance based on signal-to-noise ratios is invariant of the particular statistics of the background noise. Performance is the same (by this criterion), for background noises having identical spectra and the same average intensities.

For the correlators principally considered in our present study (involving statistically dependent background noises) we may draw the following conclusions: In the case of normal noise interference, we observe again that (from 6.4a, b) vs. (6.5a, b) vs (6.6a, b)) with deterministic signals (where $k_S(\tau) \neq 0$ as $\tau \to \infty$ and $k_S(\tau) = 1$ for suitable τ), $(S/N)^2_{out}$ of these simple auto-correlation receivers is always 3 db less than $(S/N)^2_{out}$ for the cross-correlation devices, with no or heavy smoothing, strong or weak input signals.

Next, for impulsive noise backgrounds, and mixtures,* we note that

(5) When the impulse density (γ) becomes infinite, $(S/N)^2_I$ approaches $(S/N)^2_G$, as expected, since then the interfering noise becomes normal and the impulse factor $\Lambda(\tau) \to 0$, cf. (4.9), (4.12).

(6) With strong input signals (a^2_{01}, $a^2_{01} \to \infty$), $(S/N)^2_I$ approaches $(S/N)^2_G$, for either weak, heavy, or intermediate smoothing, and both auto-and cross-correlation reception. This is easily seen from (4.9), (4.12) and the fact that $\Lambda = 0(a^4_0)$ while $(S/N)^2_G = 0(a^2_0)$ in the strong signal cases (cf. (3) above.)

(7) Cross-correlation systems can make $\Lambda^{(12)}(\tau) \to 0$ and hence eliminate the degradation due to the impulsive character of the noise, so that again $(S/N)^2_I$ approaches $(S/N)^2_G$, for all input signal levels, either by setting $a^2_{02} \to \infty$ with a finite delay (τ), or with a^2_{02} ($\gg a^2_{01}$) finite, by choosing τ infinite. Deterministic signals are necessarily required here for cross-correlation.

Of particular importance are the results that:

(8) Impulse noise is always worse than, or at best equal to, gauss noise (of equivalent spectrum and total intensity), i.e. impulse noise always degrades $(S/N)^2_I$ vis-à-vis $(S/N)^2_G$, the more so as the background becomes more impulsive, i.e. as $\gamma \to 0$ (with $N^2_{imp} > 0$).

*Equivalent in total intensity and spectral shape to a standard gauss noise, cf. (4.4) et seq.

(9) Consequently, auto-correlation systems may or may not be degraded in (S/N) performance by impulsive noise: With a deterministic $\overline{\text{nite}}$ signal source ($\psi_S < \infty$) one lets $\tau \rightarrow \infty$ and thus $\Lambda^{(11)}(\tau) \rightarrow 0$, because $\overline{k_u(\tau)^2}$, $k_{u^2}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, cf (6.7a-6.8b), with a resulting equivalent performance vis-à-vis a purely normal background). However, if the signals are non-deterministic, e.g. noise processes themselves, as is frequently the case in present acoustic applications, then $\Lambda^{(11)}(\tau) \neq 0$ for all finite τ^*, and one has thus possibly large degradations of $(S/N)_I^2$ vis-à-vis $(S/N)_G^2$.

(10) In the case where $\Lambda^{(12)} > 0$, when $(S/N)_G^2 \rightarrow \infty$, not because $a_0^2 \rightarrow \infty$, cf (6), but because of heavy smoothing ($\Delta f_e \rightarrow 0$, cf (1)), the degradation due to impulse noise becomes relatively infinite, since then

$$(S/N)_I^2 = [(1-\mu)\Lambda^{(12)}]^{-1}, \quad 1 > \mu \geq 0, \text{ as } (S/N)_G^2 \rightarrow \infty, \text{ cf. } (4.12).$$

The principal conclusions to be drawn from the foregoing depend heavily upon whether deterministic or stochastic signals are employed. In many important applications passive systems must be used for signal detection and estimation, and then in this respect reception is at the mercy of the signal source. For example, if the emitted signal belongs to a random process, e.g., is a normal noise or impulsive noise wave, then only auto-correlation reception is possible, usually with zero or small delays (τ) so that $k_S(\tau)$ (or the numerators of $(S/N)_G^2$, $(S/N)_I^2$) are maximized. With impulsive noise backgrounds considerable degradation of $(S/N)_{out}^2$ may be expected, cf.(8), (9), particularly for strong output** signals and high impulsiveness

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*Here τ must be finite, otherwise $k_S(\tau)$ would also vanish: there would effectively be no signal, either, cf. the numerators in (6.1)-(6.10) above.

**The input signal-to-noise ratios (a_0^2, a_{01}^2 etc.) may be quite small, but the effective output ratio is required to be at least moderately large 0(0.0 db or more) for effective detection. The enhancement of (S/N) is, of course, to be achieved by the correlation and smoothing processes discussed in the previous sections.

(small γ and $\therefore$ large $\Lambda^{(11)}$) cf. Figure 2 and (10) above. This effect is less pronounced for moderate and weak outputs, when $\Lambda^{(11)}$ is very large. A brief table of $(S/N)_I^2$, based on Eq. (4.9) illustrates these remarks:

$\left(\frac{S}{N}\right)_G^2$	$\Lambda^{(11)} = 10^{-1}$	10^0	10^1	10^2
-3db	-3.2db	-4.8 db	-10.8 db	- 20.1 db
0	-0.4	-3.30	-10.4	-20.0-
3	2.2	-1.8	-10.2	-20.0
10	7.0	-0.4	-10.0+	-20.0
20	9.6	0.0-	-10.0	-20.0

Table 7.1 $(S/N)_I^2$ for various $\Lambda^{(11)}$ and $(S/N)_G^2$

For instance, with moderately impulsive noise ($\Lambda^{(11)} = 10$) and somewhat weak output signal, $(S/N)_G^2 = 0.0$ db, the degradation is $-(0.0 - 10.4) = 10.4$ db, with much greater (relative) degradations for stronger outputs, eg. $(S/N)_G^2 = 20.0$db. These figures are moderated for mixtures of gauss and impulse noise, becoming less significant as the impulsive component is decreased relative to the normal one ($\mu \to 1$).

Whenever coherent operation is possible (in active systems only), then cross-correlation reception is naturally preferred, and the effects of the impulsive noise may be entirely suppressed, even for comparatively weak inputs (cf. (7)). Heavy post-correlation smoothing is also desired when possible, to enhance $(S/N)_{out}^2$, or equivalently, to lower the input signal level that can be perceived after detection. Moreover, from the point of view of received signal energy alone, all signals of equal energy are under the present criterion equivalent - no distinctions need be made in this respect between stochastic and

deterministic signal processes, since $k_S(\tau)$, $k_S^{(12)}(\tau)$ have always maximal value(s) of unity, for proper choice of delay. The effect on the background noise, however, is not the same in the strong signal cases, because of the (sxn) noise products, cf. Eqs. (6.1)-(6.1a).

In practice, we cannot expect to attain infinite delays or infinitely strong local signals ($a_{02}^2 \rightarrow \infty$), so that these ideal limiting conditions are not reached, but they can usually be well enough approximated for the analogies to be applicable. Measures of performance based on signal-to-noise ratio are, of course, incomplete, as explained in Section 1, so that interpretations employing the results of the present study are similarly limited. However, they are quantitatively useful within their proper framework and qualitatively indicative of performance within the larger statistical picture, e.g., enhancing the output signal-to-noise ratio acts to reduce the probabilities of decision error, and so on. The present treatment has purposely been kept in as canonical a form as possible in view of the many system parameters and their combinations. Also, from the general results here (cf. Sec. 6) the many special relations of concern to more detailed and local interests can be readily obtained. A later study will consider other types of non-normal noise backgrounds, and extend receiver structure to include multiple element arrays.

APPENDIX I

POISSON IMPULSE NOISE

For the specific applications of the present study we extend the results of earlier work[12] for an impulsive noise of the Poisson type, characterized by arbitrary numbers of independent impulses occurring randomly in time, in order to obtain the desired second-and fourth-order moments needed in the theory. The required generalization is that of Eq. (11.73a), ref. 1, for the joint characteristic function of $N_1^{(1)}$, $N_3^{(2)}$, $N_3^{(2)}$, $N_4^{(2)}$ (with $N_1^{(1)} = N^{(1)}(t_1)$, etc.), and is found to be

$$F_4(i\xi_1, \ldots, i\xi_4) = \exp\left[\gamma \int_0^\infty d\tau_1 \ldots \int_0^{2\pi} d\phi_2 \int dz \left(\exp\left\{ia_1 \cos\phi_1 \sum_{\ell=3,4} \xi_\ell u_1(z_\ell, \tau_1) + ia_2 \cos\phi_2 \sum_{\ell=3,4} \eta_\ell u_2(z_\ell, \tau_2)\right\} - 1\right)\right] \tag{A. 1-1}$$

where

γ = (average no. of pulses/sec.) $\cdot$ (average duration, $\bar{\tau}$, of a typical impulse) $\equiv$ impulse "density";

$z_\ell = (t_\ell - t_1)\beta^{-1}$; $\beta = \bar{\tau}^{-1}$; (A. 1-1a)

u_1, u_2 = basic pulse shapes for $N^{(1)}$, $N^{(2)}$;

$\tau_1, \tau_2, a_1, a_2, \phi_1, \phi_2$, = possible random parameters associated with the basic impulses.

(It is assumed that, in general, the mechanism generating $N^{(1)}$ and $N^{(2)}$ is a common one, so that $N^{(1)}$ and $N^{(2)}$ are correlated. This implies a suitable statistical relationship between $a_1 u_1 \cos\phi_1$ and $a_2 u_2 \cos\phi_2$, of course.)

The desired moments are found in the usual way by differentiating the characteristic function (A. 1-1). We have in the general situation when $\overline{N}$, $\overline{N_i N_j N_k} \neq 0$

$$\overline{N^{(1)}(t_1)N^{(2)}(t_4)} = - \frac{\partial^2}{\partial\xi_1 \partial\xi_4} F_4 \Bigg|_{\xi_1 = \dots = 0}$$

$$= \gamma\,\beta \left\langle A_1 A_2 \int_{-\infty}^{\infty} u_1(\tau_0 + t_1; \theta_1)\; u_2(t_4 + \tau_4; \theta_2)\, d\tau_0 \right\rangle$$

$$+ \gamma^2 \beta^2 \left\langle \int_{-\infty}^{\infty} A_1 u_1 (\tau_0 + t_1; \theta_1)\, d\tau_0 \right\rangle \left\langle \int_{-\infty}^{\infty} A_2 u_2 (\tau_0 + t_4; \theta_2)\, d\tau_0 \right\rangle , \qquad \text{(A. 1-2)}$$

where $A_1 = a_1 \cos\phi_1$, $A_2 = a_2 \cos\phi_2$, and θ_1, θ_2 refer to any other possibly random parameters of the basic waveforms u_1, u_2. The other second moments follows in similar fashion, viz.:

$$\overline{N^{(1)}(t_1)N^{(1)}(t_2)} = - \frac{\partial^2}{\partial\xi_1 \partial\xi_2} F_4 \Bigg|_{\xi_1 = \dots = 0}$$

$$= \gamma\,\beta \left\langle A_1^2 \int_{-\infty}^{\infty} u_1 (t_1 + \tau_0; \theta_1)\; u_1 (t_2 + \tau_0; \theta_1)\, d\tau_0 \right\rangle$$

$$+ (\gamma\beta)^2 \left\langle \int_{-\infty}^{\infty} A_1 u_1 (t_1 + \tau_0; \theta_1) \right\rangle^2 \quad \text{etc.} \qquad \text{(A. 1-3)}$$

The general fourth-order moment is found to be

$$I_4 = \overline{N_1^{(1)} N_2^{(1)} N_3^{(2)} N_4^{(2)}} = \left. \frac{\partial^4}{\partial \xi_1 \ldots \partial \xi_4} F_4 \right|_{\xi_1 = \ldots = 0}$$

$$= \gamma \beta \int_{-\infty}^{\infty} \langle A_1^2 A_2^2 u_1 u_2 u_3 u_4 \rangle \, d\tau_0 + \gamma^2 \beta^2 \Big\{ \int \langle A_1 A_2^2 u_2 u_3 u_4 \rangle \, d\tau_0 \langle \int A_1 u_1 d\tau_0 \rangle$$

$$+ \langle \int A_1 A_2^2 u_1 u_3 u_4 \, d\tau_0 \rangle \langle \int A_1 u_2 \rangle \, d\tau_0 + \int \langle A_1^2 A_2 u_1 u_2 u_4 \rangle \, d\tau_0 \langle \int A_2 u_3 d\tau_0 \rangle$$

$$+ \langle \int A_1^2 A_2 u_1 u_2 u_3 d\tau_0 \rangle \langle \int A_2 u_4 \rangle \, d\tau_0 \Big\} + \gamma^2 \beta^2 \Big\{ \int \langle A_2^2 u_3 u_4 \rangle d\tau_0 \int \langle A_1^2 u_1 u_2 \rangle d\tau_0$$

$$+ \int \langle A_1 A_2 u_2 u_4 \rangle d\tau_0 \int \langle A_1 A_2 u_1 u_3 \rangle d\tau_0 + \int \langle A_1 A_2 u_1 u_4 \rangle d\tau_0 \int \langle A_1 A_2 u_2 u_3 \rangle d\tau_0 \Big\}$$

$$+ \gamma^3 \beta^3 \Big\{ \int \langle A_2^2 u_3 u_4 \rangle d\tau_0 \int \langle A_1 u_2 \rangle d\tau_0 \int \langle A_1 u_1 \rangle d\tau_0$$

$$+ \int \langle A_1 A_2 u_2 u_4 \rangle \, d\tau_0 \int \langle A_2 u_3 \rangle d\tau_0 \int \langle A_1 u_1 \rangle d\tau_0$$

$$+ \int \langle A_1 A_2 u_2 u_3 \rangle d\tau_0 \int \langle A_1 u_1 \rangle \, d\tau_0 \int \langle A_2 u_4 \rangle \, d\tau_0$$

$$+ \int \langle A_1 A_2 u_1 u_4 \rangle d\tau_0 \int \langle A_2 u_3 \rangle d\tau_0 \cdot \int \langle A_1 u_2 \quad d\tau_0 \rangle$$

$$+ \int \langle A_1 A_2 u_1 u_3 \rangle d\tau_0 \int \langle A_1 u_2 \rangle d\tau_0 \int \langle A_2 u_4 \rangle d\tau_0$$

$$+ \int \langle A_1^2 u_1 u_2 \rangle d\tau_0 \int \langle A_2 u_3 \rangle d\tau_0 \int \langle A_2 u_4 \rangle d\tau_0 \Big\}$$

$$+ \gamma^4 \beta^4 \Big\{ \langle \int A_1 u_1 \rangle d\tau_0 \int \langle A_2 u_2 \rangle d\tau_0 \int \langle A_3 u_3 \rangle d\tau_0 \int \langle A_4 u_4 \rangle d\tau_0 \Big\}, \tag{A. 1-4}$$

where now our abbreviated notation is $u_1 = u_1(t_1 + \tau_0; \theta_1)$, $u_2 = u_1(t_2 + \tau_0; \theta_1)$; $u_3 = u_2\,(t_3 + \tau_0; \theta_2)$, $u_4 = u_2\,(t_4 + \tau_0; \theta_2)$. For the important case considered here specifically, the first and third moments of N vanish, e. g., $\overline{N}_i = 0$; $\overline{N_i N_j N_k} = 0$, so that I_4 reduces at once to the much simpler form

$$I_4 = \gamma\beta \int \langle A_1^2 A_2^2 u_1 u_2 u_3 u_4 \rangle d\tau_0 + \gamma^2\beta^2 \Big\{ \int \langle A_2^2 u_3 u_4 \rangle d\tau_0 \int \langle A_1^2 u_1 u_2 \rangle d\tau_0$$

$$+ \int \langle A_1 A_2 u_2 u_4 \rangle d\tau_0 \int \langle A_1 A_2 u_1 u_3 \rangle d\tau_0 + \int \langle A_1 A_2 u_1 u_4 \rangle d\tau_0 \int \langle A_1 A_2 u_2 u_3 \rangle d\tau_0 \Big\}. \tag{A. 1-5}$$

Accordingly, with this assumption of the vanishing of the odd-order moments of N, we have with the help of (A. 1-5) the following expression for $\left[A_4^{(12}(\tau, x)_N - K_N^{(12)}(\tau)^2\right]_{imp}$, cf. (3. 10), in these stationary cases:

$$\left[A_4^{(12)}(\tau, x)_N - K_N^{(12)}(\tau)^2\right]_{imp}$$

$$= \beta\gamma\int_{-\infty}^{\infty}\left\langle A_1^2 A_2^2 u_1(\tau_0 + \tau)\, u_1(\tau_0 + \tau + x)\, u_2(\tau_0)\, u_2(\tau_0 + x)\right\rangle d\tau_0$$

$$+ \gamma^2\beta^2\left\{\int_{-\infty}^{\infty}\left\langle A_1^2 u_1(\tau_0)\, u_1(\tau_0 + x)\right\rangle d\tau_0 \cdot \int_{-\infty}^{\infty}\left\langle A_1^2\, u_2(\tau_0)\, u_2(\tau_0 + \tau)\right\rangle d\tau_0\right.$$

$$\left. + \int_{-\infty}^{\infty}\left\langle A_1 A_2 u_1(\tau_0 + \tau - x)\, u_2(\tau_0)\right\rangle d\tau_0 \cdot \int_{-\infty}^{\infty}\left\langle A_1 A_2 u_1(\tau_0 + \tau + x)\, u_2(\tau_0)\right\rangle d\tau_0\right\} . \tag{A. 1-6}$$

Since

$$\beta\,\gamma\int_{-\infty}^{\infty}\overline{A_1^2 u_1(\tau_0)\, u_1(\tau_0 + x)}\, d\tau_0 \;\equiv\; K_N^{(1)}(x)_{imp} ,$$

$$\beta\,\gamma\int_{-\infty}^{\infty}\overline{A_2^2 u_2(\tau_0)\, u_2(\tau_0 + x)}\, d\tau \;\equiv\; K_N^{(2)}(x)_{imp} ,$$

$$\beta\,\gamma\int_{-\infty}^{\infty}\overline{A_1 A_2\, u_1(\tau_0 + \tau - x)\, u_2(\tau_0)}\, d\tau_0 \;\equiv\; K_N^{(12)}(\tau - x)_{imp} , \tag{A. 1-7}$$

$$\beta\,\gamma\int_{-\infty}^{\infty}\overline{A_1 A_2\, u_1(\tau_0 + t + x) u_2(\tau_0)}\, d\tau_0 \equiv\; K_N^{(12)}(\tau + x)_{imp} ,$$

and*

$$\gamma\beta \int_{-\infty}^{\infty} \overline{A_1^2 A_2^2\, u_1(\tau_0+\tau)\, u_1(\tau_0+\tau+x)\, u_2(\tau_0)\, u_2(\tau_0+x)}\; d\tau_0 \equiv I_{12}(\tau, x), \qquad \text{(A. 1-8)}$$

it follows that

$$\left[A_4^{(12)}(\tau, x)_N - K_N^{(12)}(\tau)^2\right]_{imp}$$

$$= I_{12}(\tau, x) + \left\{ K_N^{(1)}(x) K_N^{(2)}(x) + K_N^{(12)}(\tau - x)\, K_N^{(12)}(\tau + x) \right\}_{imp}, \qquad \text{(A. 1-9)}$$

which is just (4. 3), (4. 3a).

*Eq. (A. 1-8) is the generalization of Eq. (66) of Magness.[8]

APPENDIX II

Proof that $\Lambda^{(12)}(\tau)_I \geq 0$

Essential to our explicit comparisons of receiver performance, and to the general conclusions of Section 7, is the requirement that $\Lambda^{(12)}(\tau)_I$, (4. 9), be positive (or zero). We demonstrate this below.

From (4. 9) we may write

$$\Lambda^{(12)}(\tau)_I = B^2(\tau) \int_{-\infty}^{\infty} \rho_T(x)\,dx \int_{-\infty}^{\infty} \overline{u_1(\lambda+\tau)\, u_1(\lambda+\tau+x)\, u_2(\lambda)\, u_2(\lambda+x)}\, d\lambda\,, \qquad \text{(A. 2-1)}$$

so that to establish that $\Lambda^{(12)}(\tau)_I \geq 0$ we must show that $I_{12}(\tau)$ $\left(\sim \int_{-\infty}^{\infty} I_{12}(\tau, x) \cdot \rho_T(x)\,dx, \text{ cf. (4. 3a)} \right)$, the coefficient of $B^2(\tau)$ in (A. 2-1), is positive or zero. We start with the Fourier transforms of the basic impulses

$$u_1(t) = \int_{-\infty}^{\infty} S_1(i\omega)\, e^{i\omega t}\, df;\; u_2(t) = \int_{-\infty}^{\infty} S_2(i\omega)\, e^{i\omega t}\, df; \quad \omega = 2\pi f \qquad \text{(A. 2-2)}$$

and write

$$I_{12}(\tau) = \int_{-\infty}^{\infty} \rho_T(x)\,dx \left\langle \int_{-\infty}^{\infty} d\lambda \int \cdots \int_{-\infty}^{\infty} S_1(i\omega)\, e^{i\omega(\lambda+x)} S_1(i\omega')\, d^{i\omega'(\lambda+x+\tau)} \right.$$

$$\left. \cdot\; S_2(i\omega'')\, e^{i\omega''\lambda} \cdot S_2(i\omega''')\, e^{i\omega'''(\lambda+x)} df \,..\, df''' \right\rangle . \qquad \text{(A. 2-3)}$$

Using $\int_{-\infty}^{\infty} \exp i(\omega + \omega' + \omega'' + \omega''')\, \lambda \, d\lambda = \delta(f''' + f'' + f)$ and making the transformations $\omega'' + \omega = \hat{\omega}$, $\omega' = -\omega'$ we may regroup and rewrite (A. 2-3) finally as

$$\hat{I}_{12}(\tau) = \int_{-\infty}^{\infty} \rho_T(x)\, dx$$

$$\cdot \left\langle \iiint_{-\infty}^{\infty} \left[S_1(i\omega) S_2(i\hat{\omega} - i\omega) e^{i\omega\tau} \right] \left[S_1(i\hat{\omega}') S_2(i\hat{\omega} - i\hat{\omega}') e^{i\hat{\omega}'\tau} \right]^* e^{-i\hat{\omega} x} df \, d\hat{f} \, d\hat{f}' \right\rangle \quad \text{(A. 2-4)}$$

Next, we define

$$G_{12}(i\hat{\omega}_i \tau) \equiv \int_{-\infty}^{\infty} S_1(i\omega) S_2(i\hat{\omega} - i\omega)\, e^{i\omega\tau} \, df. \quad \text{(A. 2-5)}$$

Accordingly, (A. 2-4) becomes

$$\hat{I}_{12}(\tau) = \int_{-\infty}^{\infty} \rho_T(x)\, dx \left\langle \int_{-\infty}^{\infty} G_{12}(i\hat{\omega};\tau)\, G_{12}(i\hat{\omega};\tau)^* \, e^{-i\hat{\omega}x} \, d\hat{f} \right\rangle . \quad \text{(A. 2-6)}$$

But we have

$$\int_{-\infty}^{\infty} \rho_T(x)\, e^{-i\hat{\omega}x} dx = \left| Y_T(i\hat{\omega}) \right|^2 , \quad \text{(A. 2-7)}$$

---

*A related result is achieved through a somewhat different approach by Magness, ref. 8, III, and Eq. (68) therein.

from pp. 163, 682 (Eq. 16.96), ref. 1, where Y_T is the system function of the truncated smoothing filter h_T, cf. (3.7a-c), so that $I_{12}(\tau)$ may be expressed as

$$\hat{I}_{12}(\tau) = \int_{-\infty}^{\infty} |Y_T(i\hat{\omega})|^2 \quad \overline{|G_{12}(i\hat{\omega};\tau|^2} \quad d\hat{f} \geq 0, \tag{A.2-8}$$

which establishes our statement above;* $I_{12}(\tau)$ vanishes only if G_{12} vanishes, all f.

Note that (A.2-5) can be represented equivalently by

$$G_{12}(i\hat{\omega};\tau) = \int_{-\infty}^{\infty} u_1(t)\, u_2(t-\tau)e^{-i\hat{\omega}(t-\tau)}\, dt = \int_{-\infty}^{\infty} u_1(t+\tau)u_2(t)e^{-i\hat{\omega}t}dt, \tag{A.2-9}$$

so that if $\tau \to \pm\infty$, then $G_{12} \to 0$, inasmuch as u_1 will vanish when u_2 is different from zero and vice-versa: there is then no essential cross-correlation between $u_1(t+\tau)$ and $u_2(t)\, e^{-i\omega\tau}$. With the help of (A.2-8) we can write $\Lambda^{(12)}(\tau)_{imp}$, (4.9) alternatively as

$$\Lambda^{(12)}(\tau)_{imp} = a_{01}^{-2} \quad a_{02}^{-2}\, k_S^{(12)}(\tau)^{-2} \quad \frac{A_1^2\, A_2^2}{\overline{A_1^2} \cdot \overline{A_2^2}}$$

$$\cdot \frac{1}{\gamma} \left\{ \frac{\int_{-\infty}^{\infty} |Y_T(i\omega)|^2 \quad \overline{|G_{12}(i\omega;\tau)|^2} \quad df}{\beta |Y_T(0)|^2 \quad \overline{G_{11}(0,0)}^2} \right\} \geq 0, \tag{A.2-10}$$

where

$$Y_T(i\omega) = \int_{0-}^{T+} h_T(\tau)\, d^{-i\omega\tau} d\tau. \tag{A.2-10a}$$

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*See Footnote, p. 47.